The Cantor Set and Banach-Tarski Paradox

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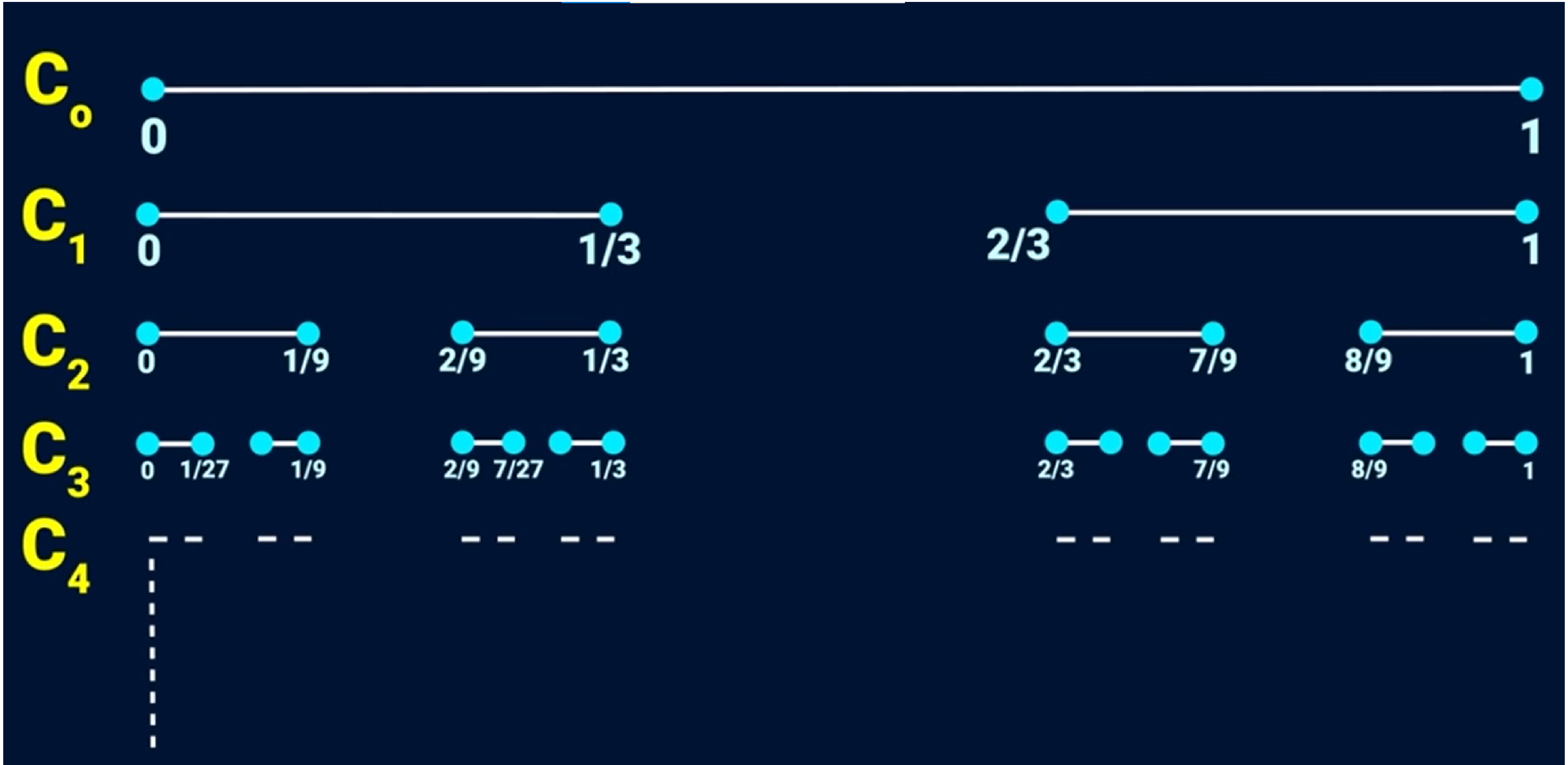
Outline

- How could a set be Lebesgue measurable while it is not Borel measurable?
    - The Cantor set
- How could a set be non-measurable?
    - Banach-Tarski Paradox
    - The Axiom of Choice

Generating the Cantor Set

- 1. Take the [0, 1] interval
- 2. Remove the middle one-third of the interval

- 3. Continue removing the middle one third of all generated line segments



(Zach Star., 2020)

The Cantor Set has a Zero Lebesgue Measure

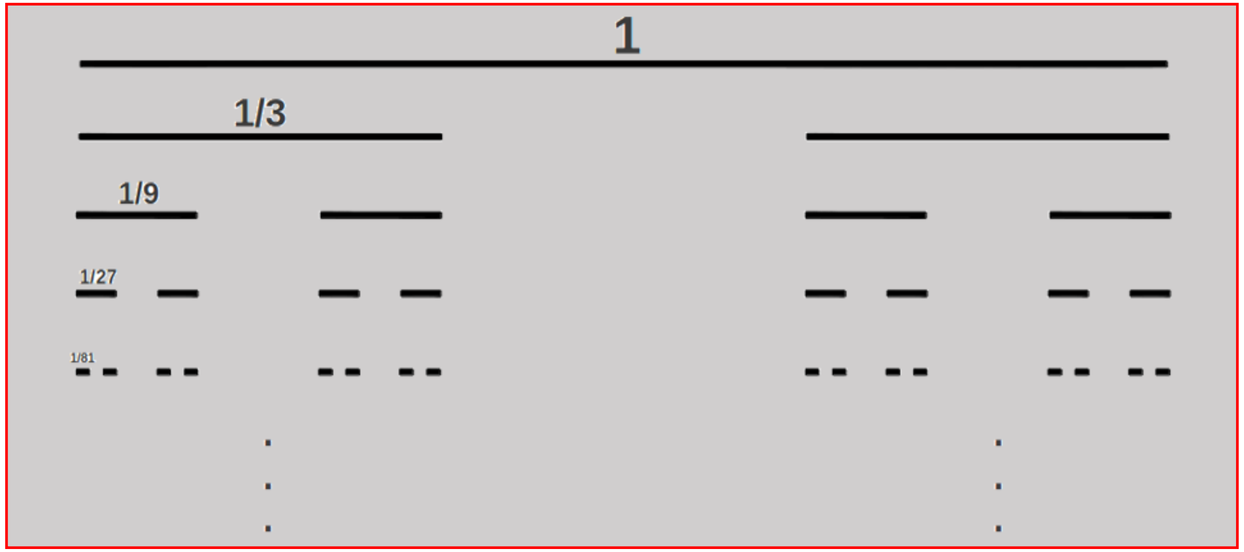
The length of each interval in C_0 is 1

The Length of each interval in C_1 is 1/3

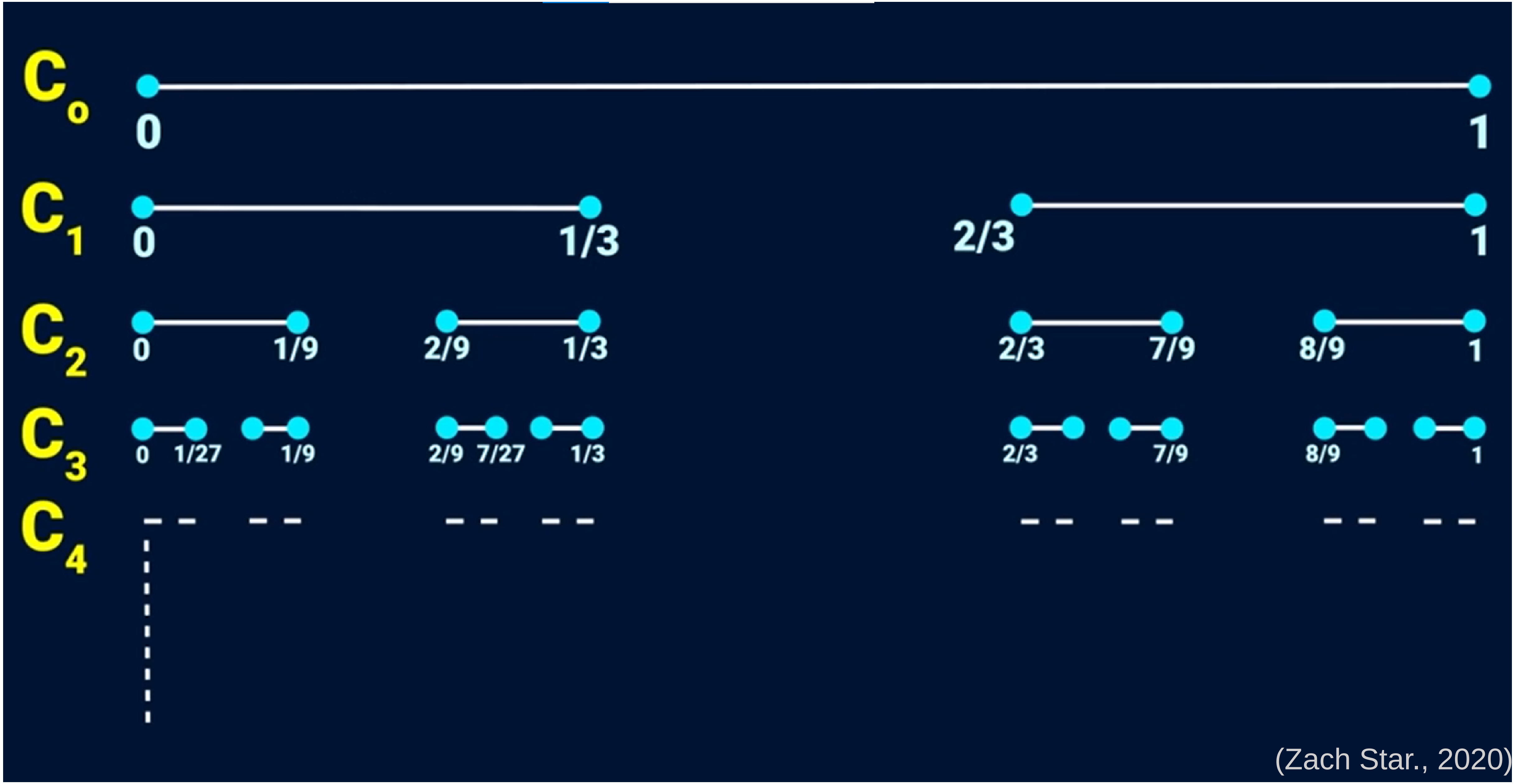
The Length of each interval in C_2 is 1/9

The Length of each interval in C_3 is 1/27

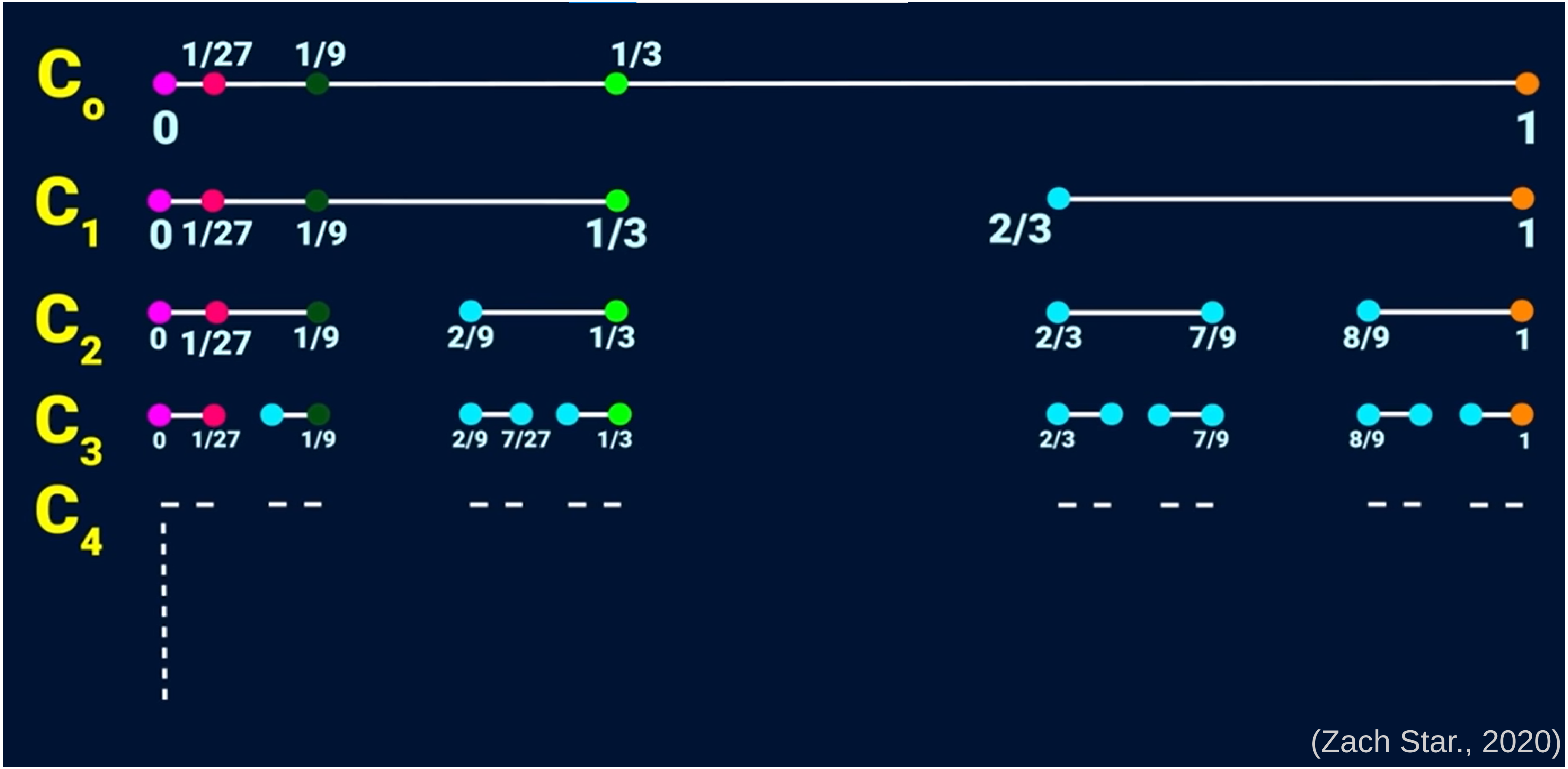
The Length of each interval in C_n is (1/3)^n



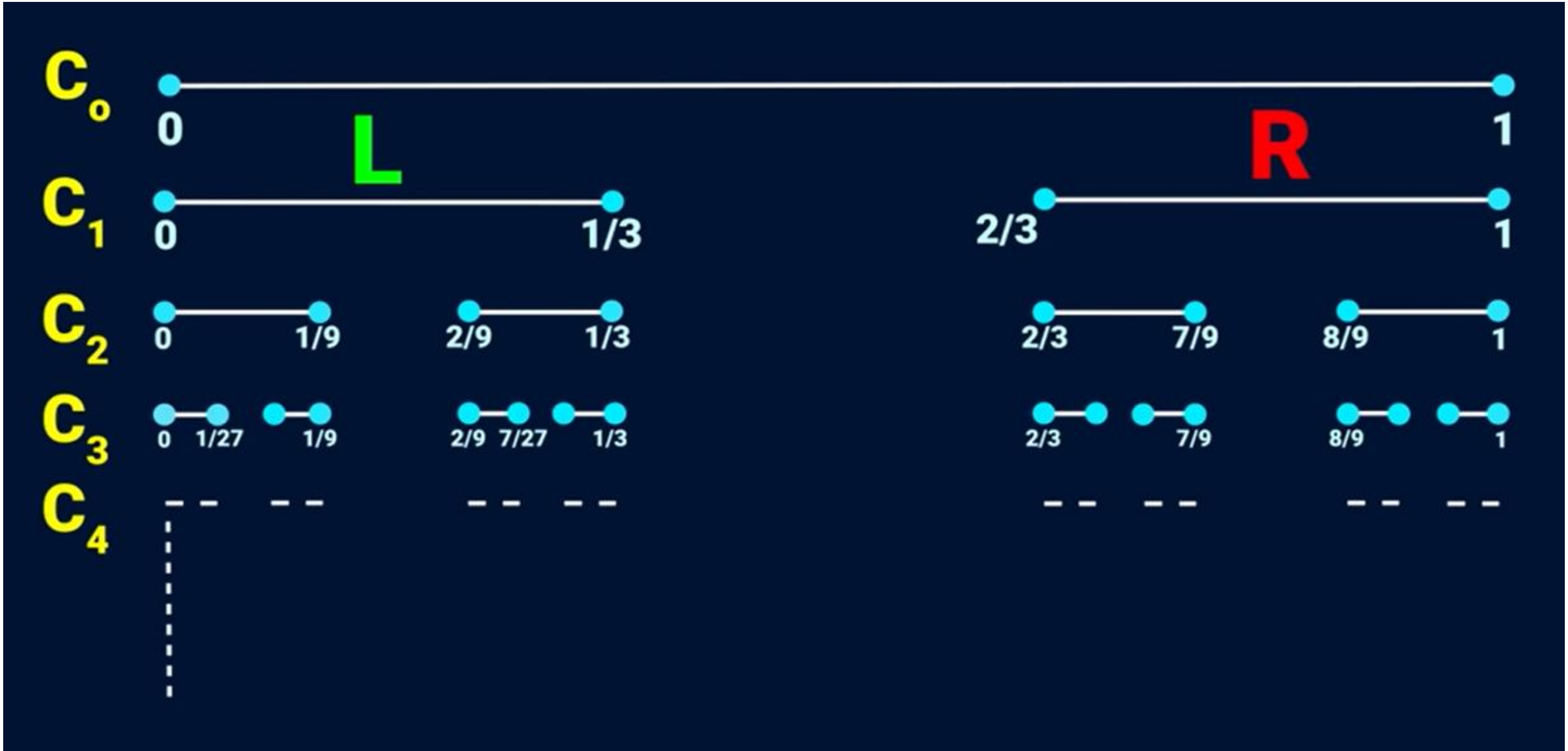
As we continue this process to infinity, we would only be left with a large number of intervals with 0 length → We will be left with a large number of points, each with a Lebesgue measure of zero → By additivity property, we observe that the Lebesgue measure of Cantor set is zero.



Cantor Set = {Points that don’t get removed}



Cantor Set = {All boundary points generated} = {0, …, 1/27, ….., 1/9, …., 8/9, …., 1}



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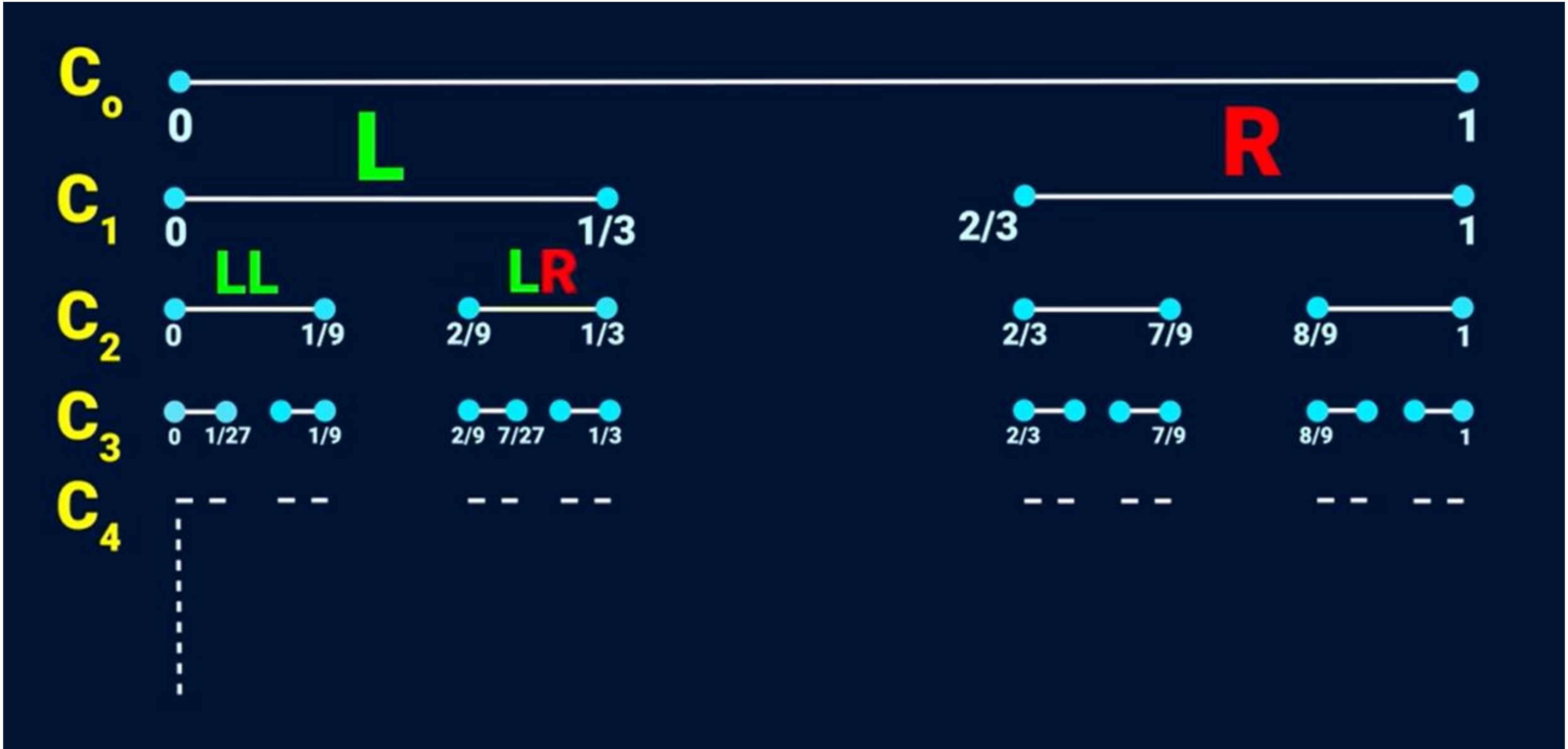

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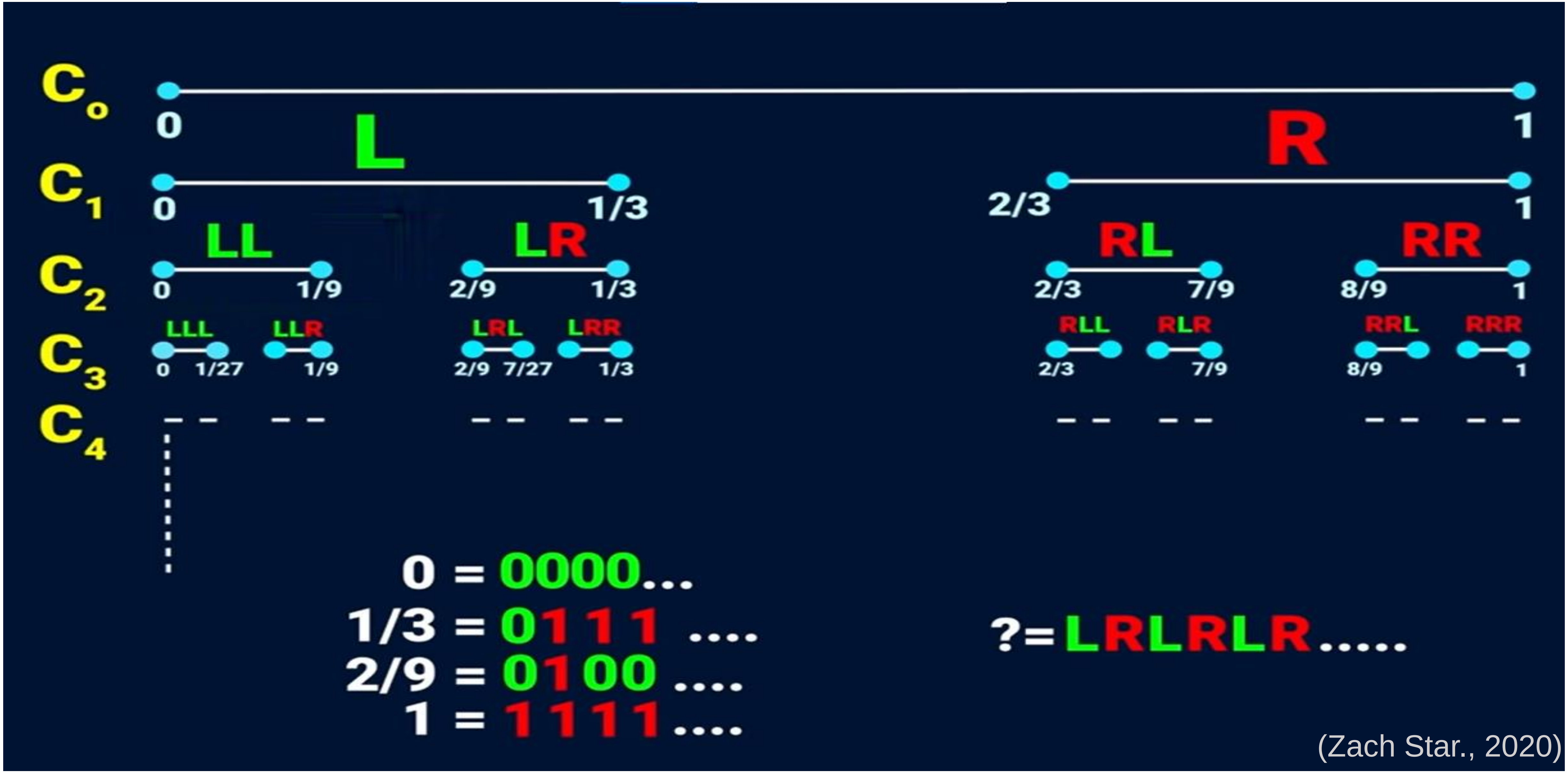
Cantor Set = {The set of all infinitely long combinations of L and R}



Cantor Set = {The set of all infinitely long binary numbers}

The Cantor Set is Uncountable

Generate a set of as many infinite binary numbers as you could

The 1st member 000000….. → flip the 1st digit
The 2nd member 000001….. → flip the 2nd digit
The 3rd member 000010….. → flip the 3rd digit
The 4th member 000100….. → flip the 4th digit
The 5th member 001000….. → flip the 5th digit
The 6th member 010000….. → flip the 6th digit
⋮

- Are these all infinite binary numbers? No. We can generate a new number: The nth number has its nth digit flipped in this new number: 111011…….

The Cantor Set is Uncountable

- Now, we are 100% sure that this new number (111011…….) is not included in our initial set of infinite binary numbers. Because we made sure that it has at least 1 different digit from any of the members that were previously included in our set!
- e.g. If someone says this new number might be the same at the 102th number of our original set, we say that our number has a different 102th digit from the 102th member of the original set. Thus, they could not be equal.

The Cantor Set is Uncountable

- Every countable set could be shown as a map from Natural numbers → any countable set has a cardinality equal to Natural numbers
- The Cantor set has a higher cardinality than Natural numbers:
  - Consider a map from Cantor set to N. Then note that we can always find a new member of Cantor set that was not mapped into N. Thus:

$$|\text{Cantor set}| > |N|$$

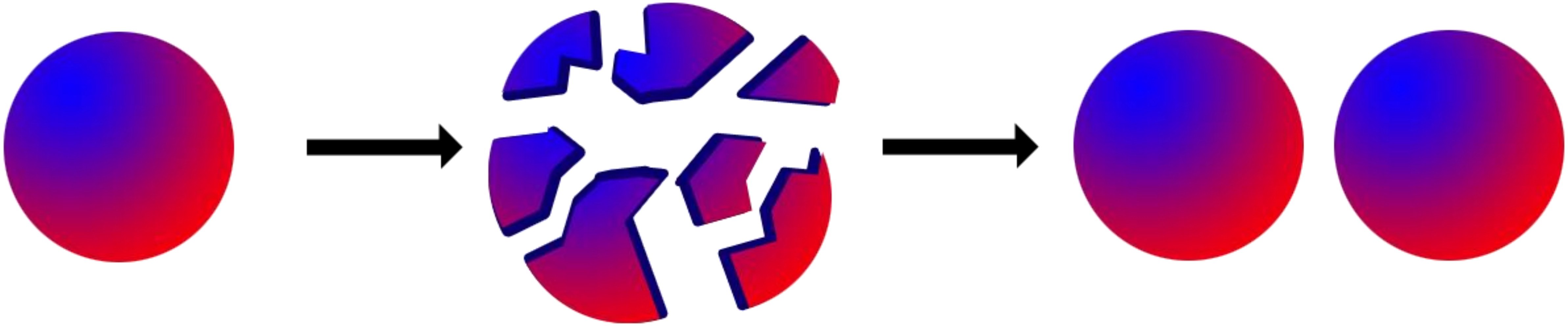
Thus, the Cantor set is uncountably infinite.

- Cantor set has Lebesgue measure zero → it consists of singletons
- → Cantor set is uncountable and we fail to find a proper pattern to write it down
- → Cantor set could not be written down as a combination of half open intervals

- *Thus, The Cantor set is not Borel Measurable, while it is Lebesgue Measurable.*

Banach-Tarski paradox

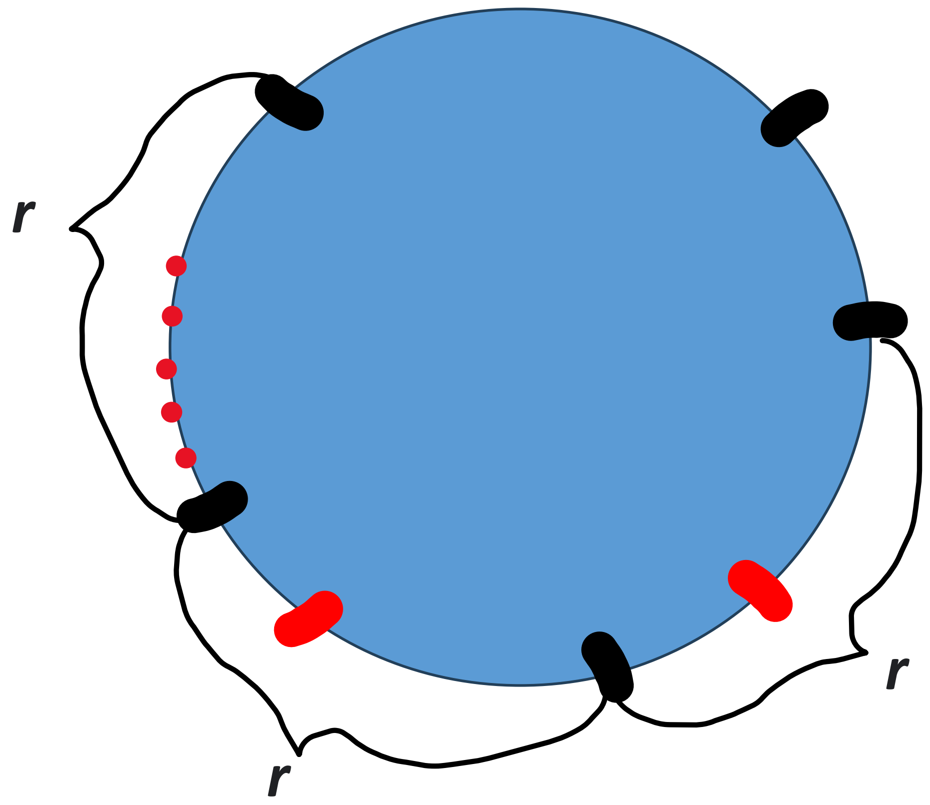
- It is always possible to carefully break a sphere into a finite number of sets such that when we take the union of those sets, we are left with two new spheres, each identical to our initial sphere. i.e Same density, size, volume, everything…



(Vsauce., 2015).

Taking an infinite number of points from the circumference of a circle

- The circumference of a circle is $2\pi r$ where 2π is an irrational number multiplied by *r*:
- → there exists no natural number that, multiplied with *r*, results in the circumference of a circle .
- → if we choose a point on a circle and move on the circle by *r* length steps, we will never walk on the same point twice!
- → thus, we can take infinitely many number of steps on a circle and at the end we will get the infinite set containing the points on the circumference of the circle.
- → Note, even if we remove 1 point from the circle, it won't effect our set when we're dealing with infinity. We can just shit our circle in place a little.



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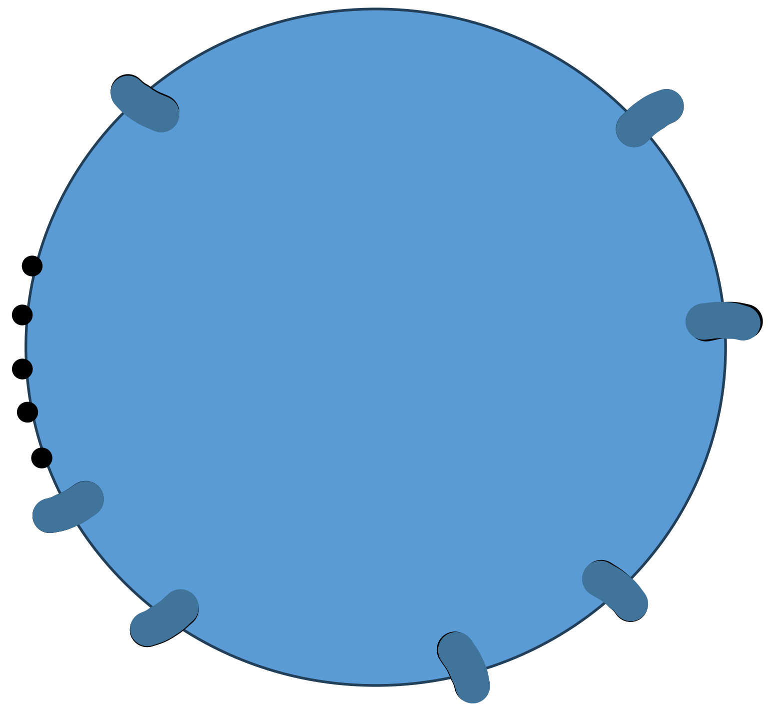
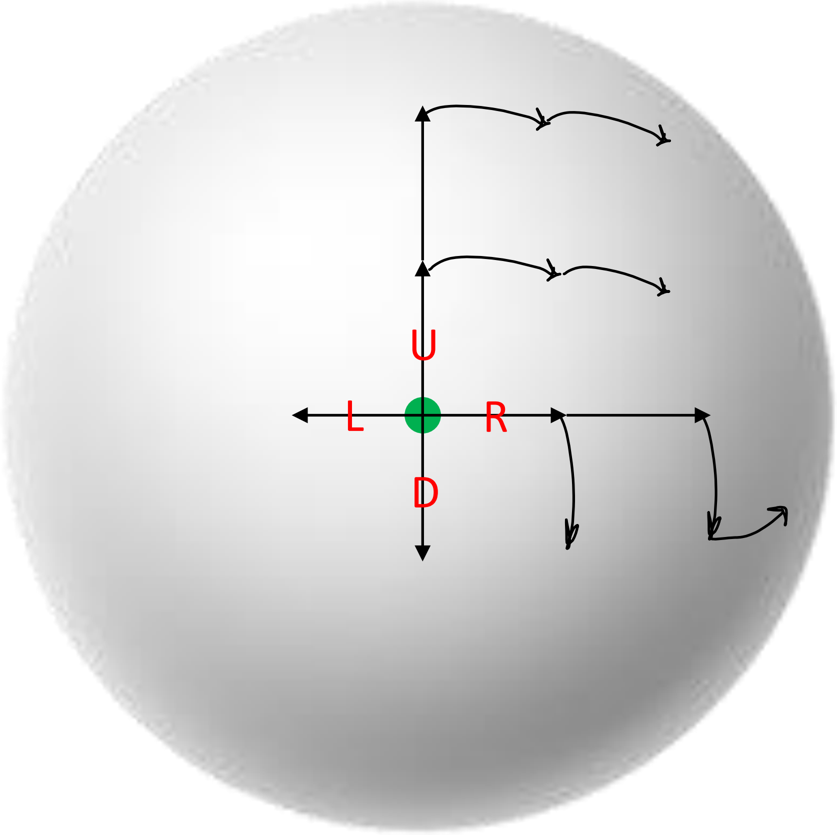
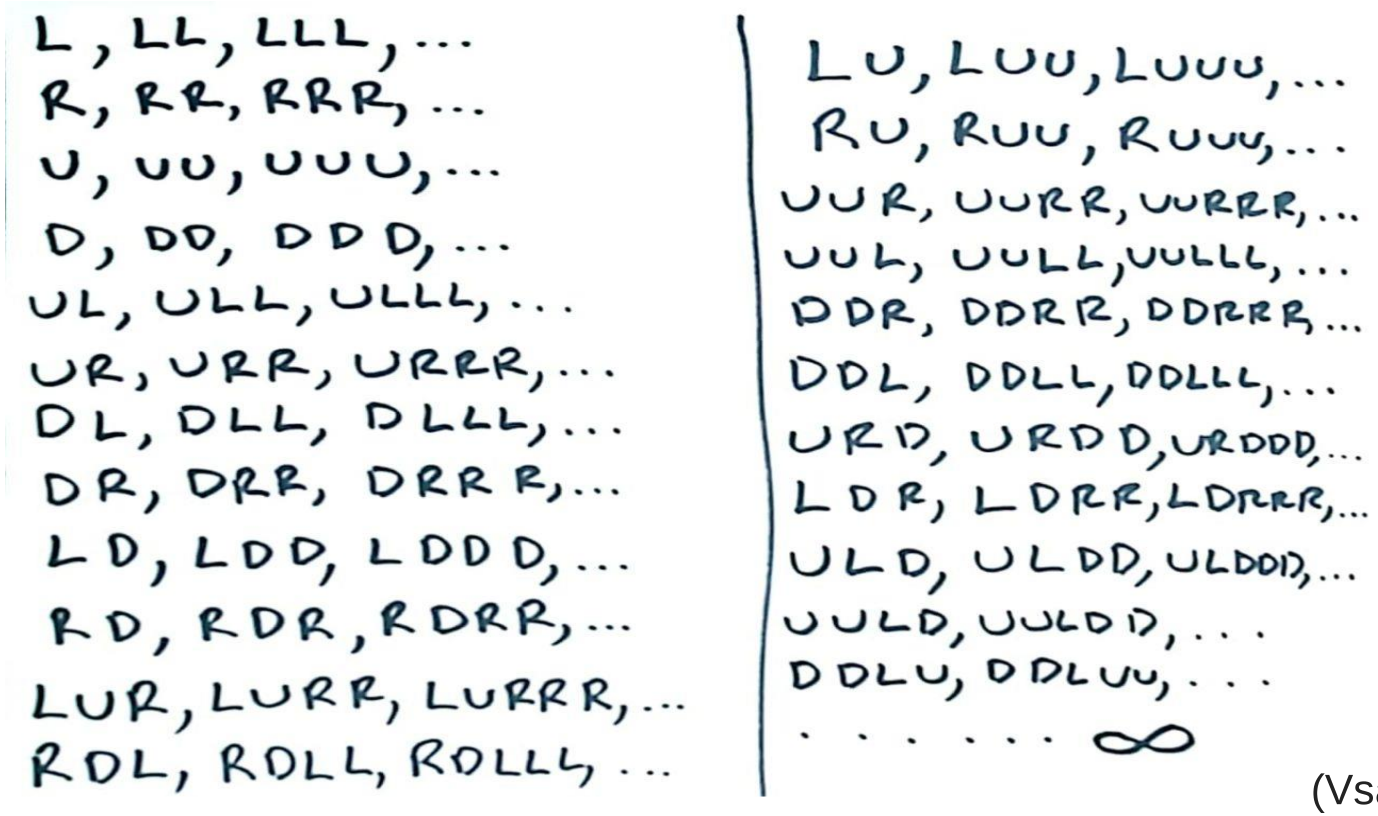


Turning a sphere into a set

- 1. Select a starting point
- 2. move up(U), down(D), left(L), and right(R) by
- 3. Make sure to avoid movements of UD, DU, LR, RL. In this way, it would be impossible to return to a point after we leave it.
- 4. Everything works similar to the previous examples: by moving through all combinations of movements possible (between U, D, L, R) and taking infinitely many points from the sphere, we will have a set of points that represent the sphere.
- 5. Each point of our set would have indices generated by the directions we need to move to get to that point. Like ULDLLLDDDDRU
- 6. Take an (uncountably) infinitely many such starting points and continue…



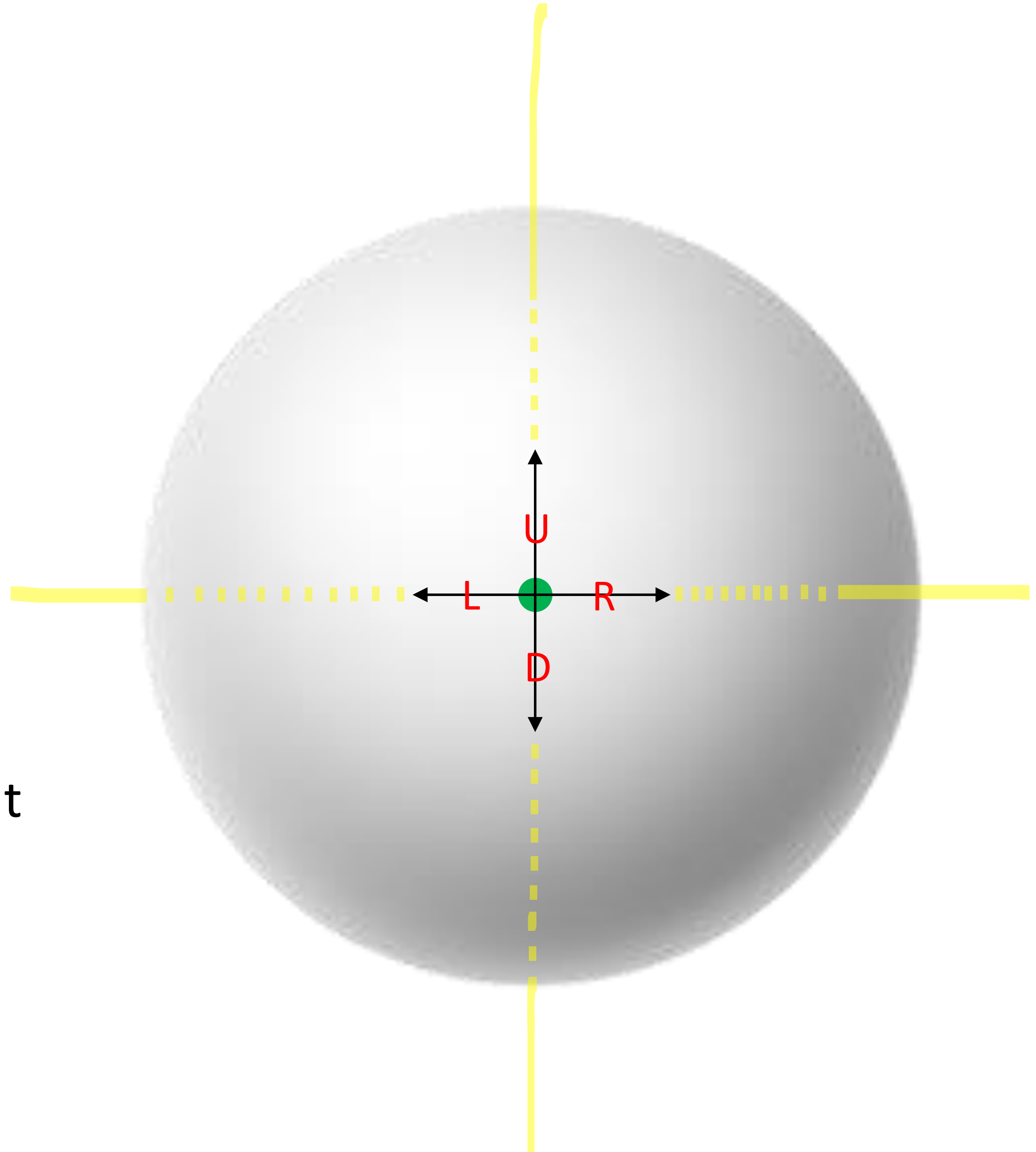
- Now we are ready to tear the entire sphere apart!

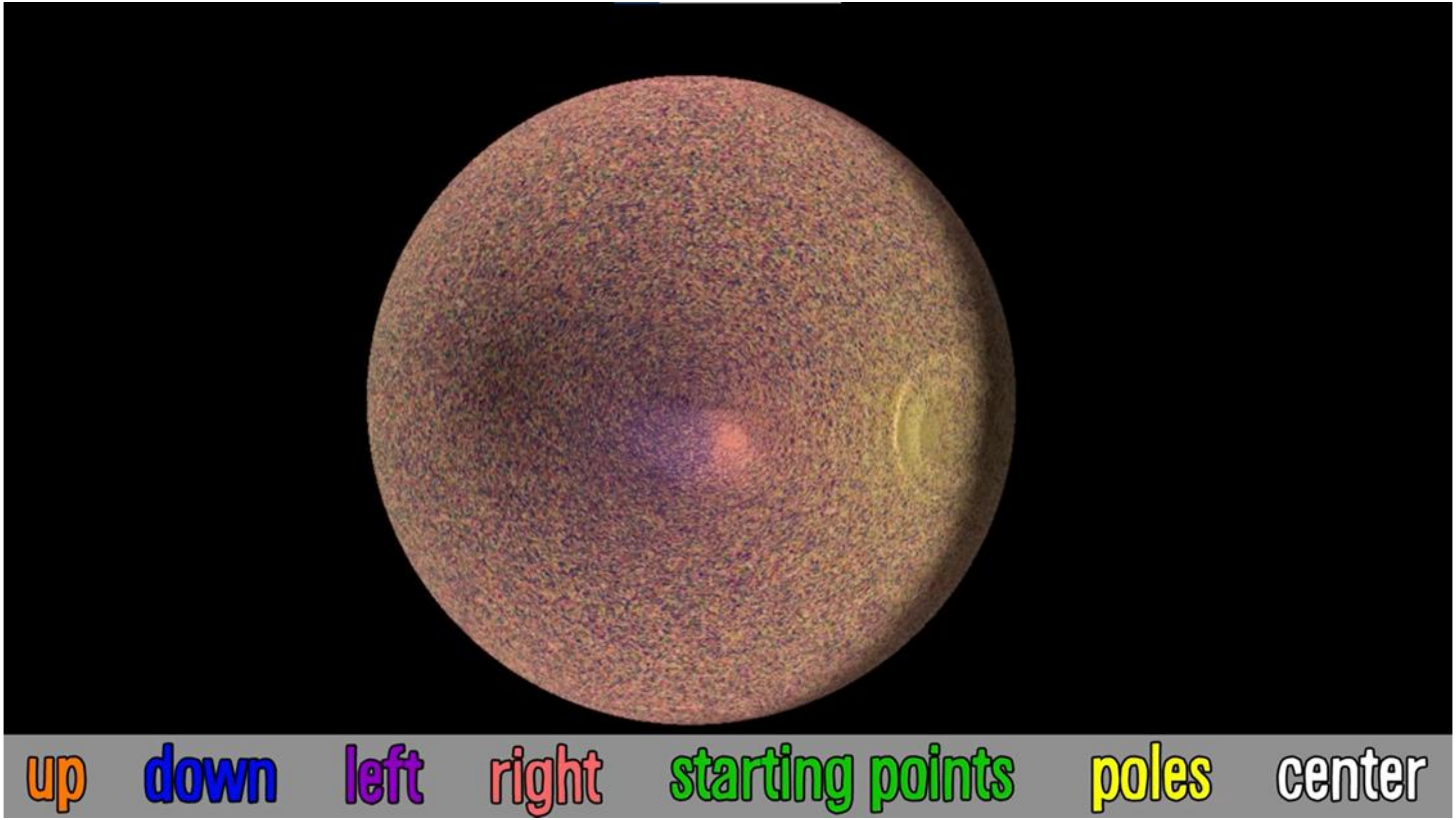


(Vsauce., 2015).

Poles and the Center point

- Now, we have exactly each point of the sphere exactly once in our set.
- → That is, other than the poles of rotation: each starting point has 2 such poles where we enter them more than once:
- → Since these are the poles of rotation, it is possible to move from both left and right (and up and down) and still reach the same points twice!
- → So we just ignore these points by removing them all and putting them in a single set that won't be used later on.
- → We also remove the center point for the same reason





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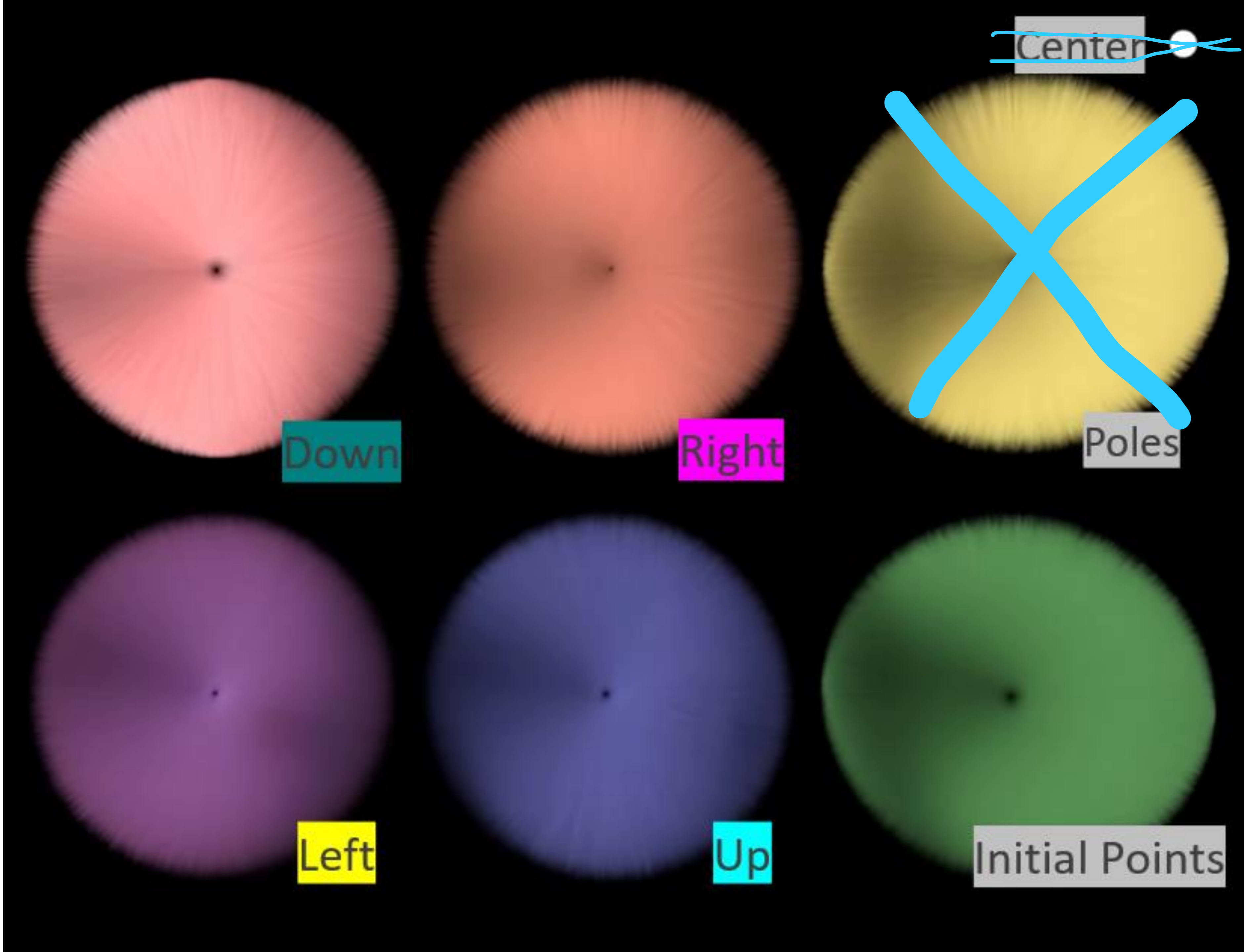
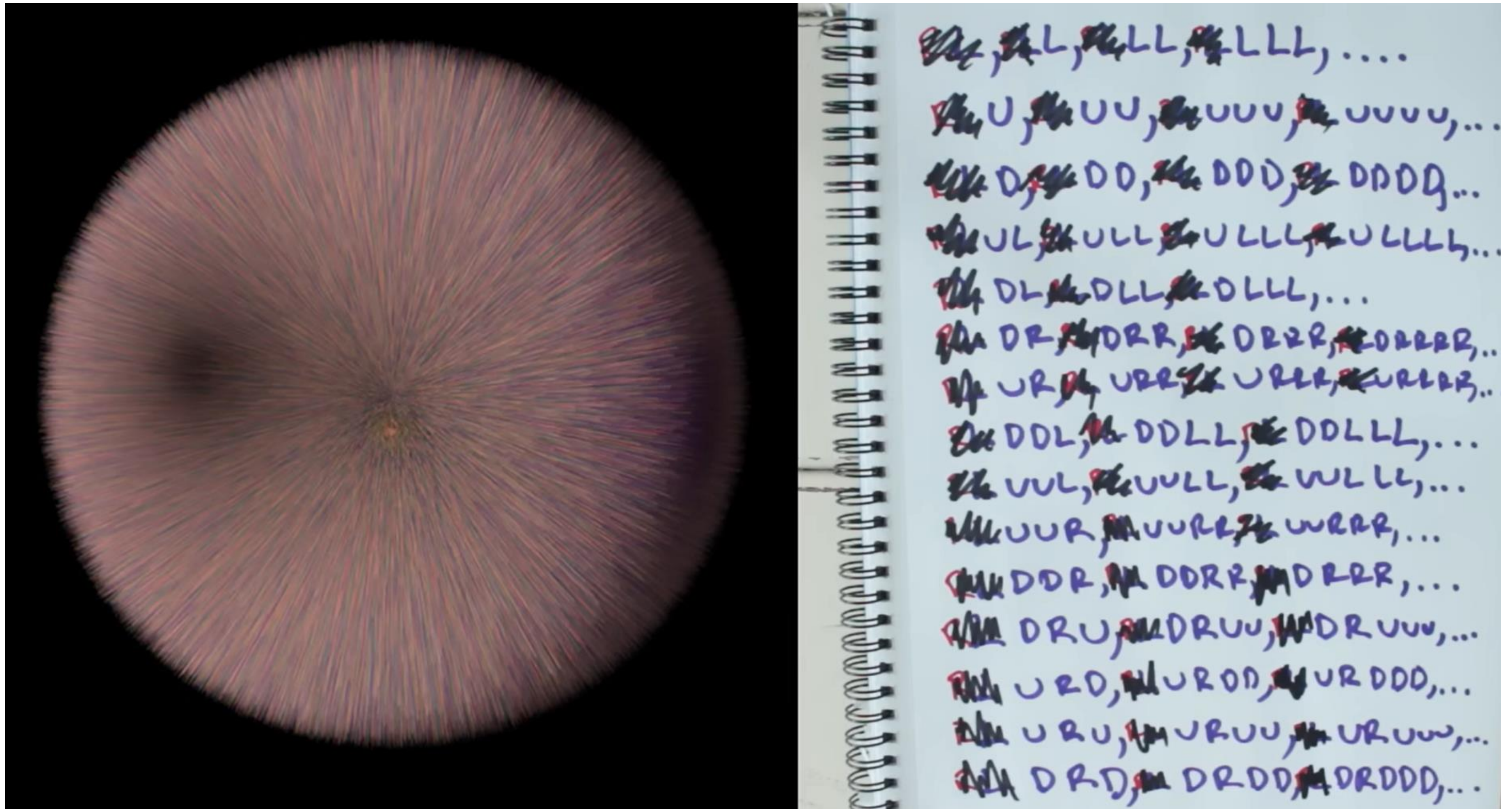
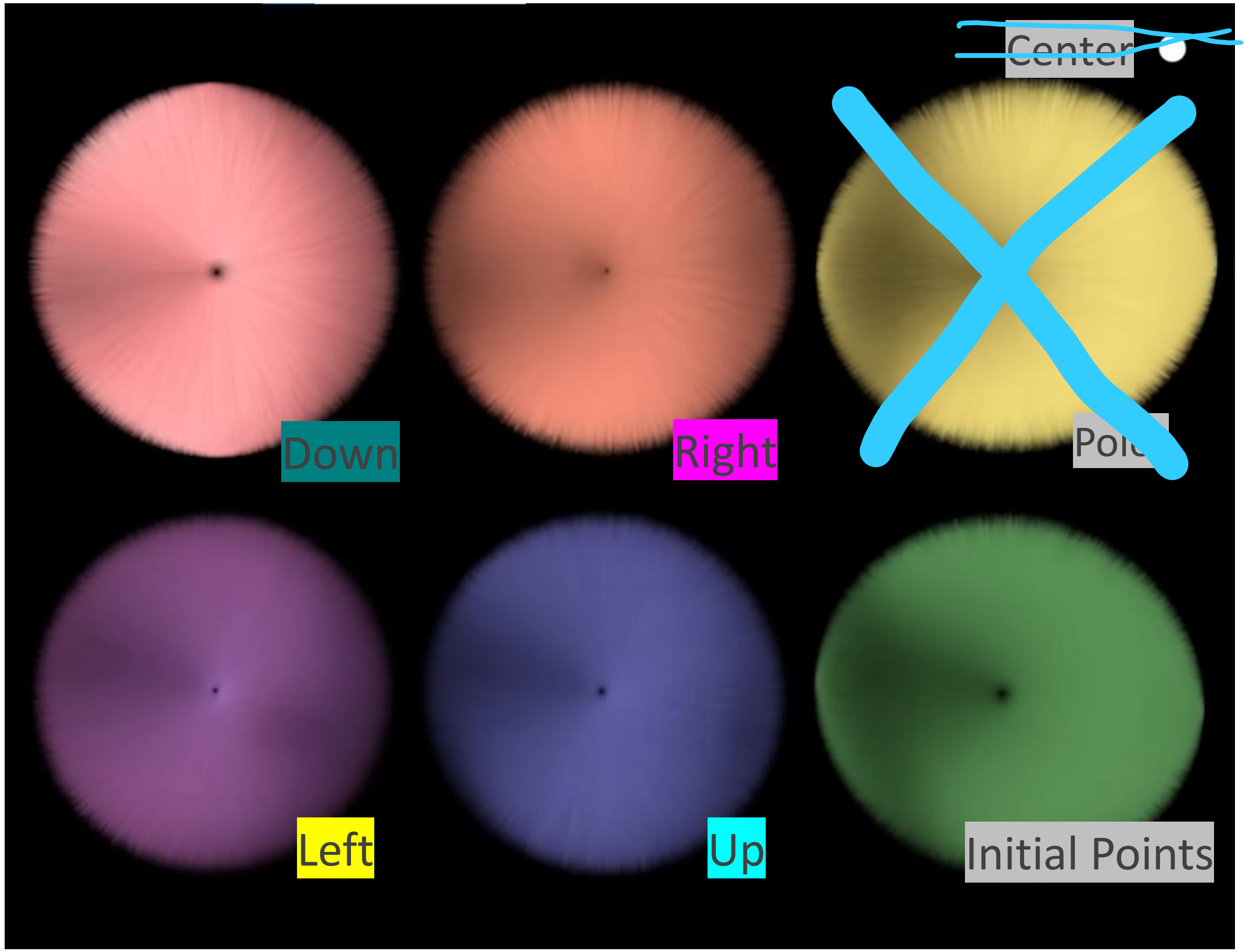

Center
Down
Right
Poles
Left
Up
Initial Points

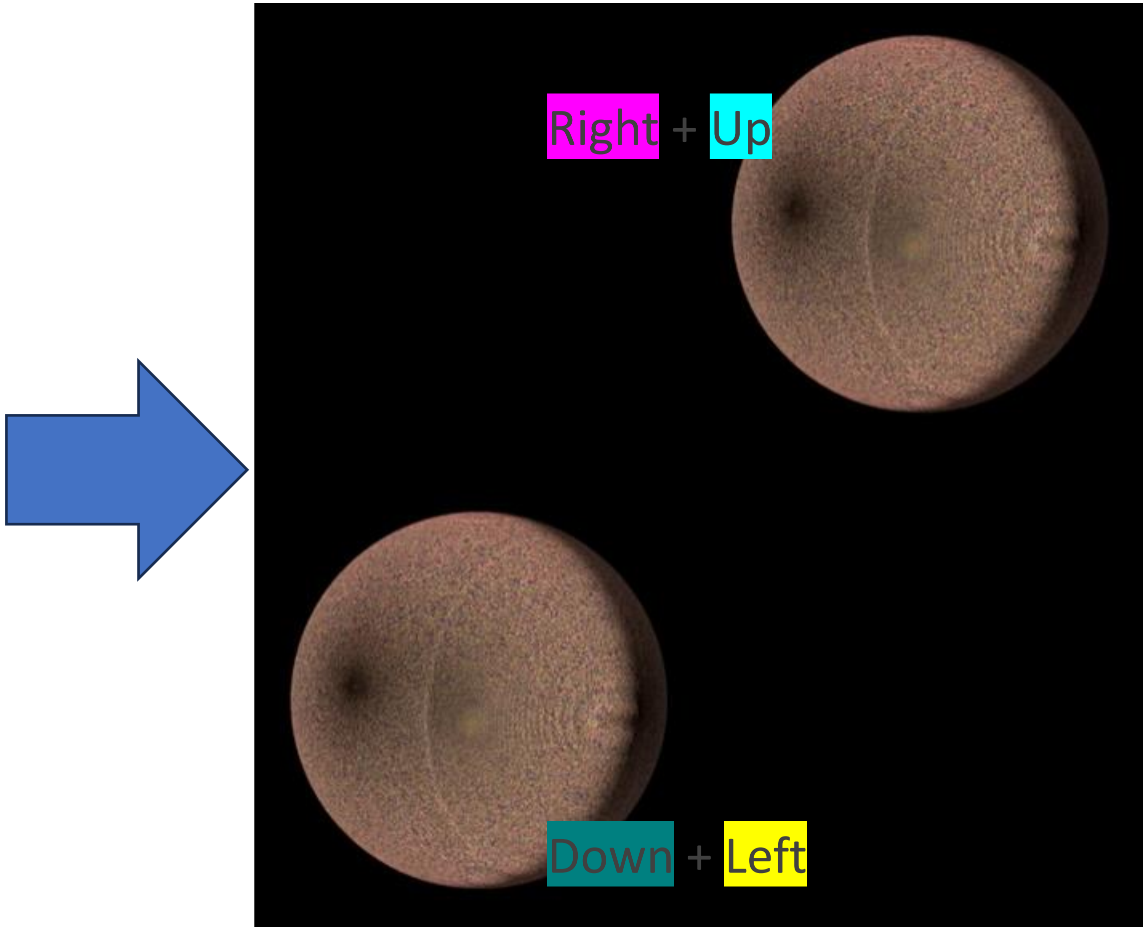
Now, we do a right rotation to the left set. The points themselves do not change because we are merely doing a rotation. But observe that the RL movement cancel itself. Thus our set would transform into a set that contains all the sets up, down, left, and starting points!



(Vsauce., 2015).

- In fact, the same situation happens when
  - we move the up set down → we'll be left with up, right, left, and starting points sets
  - we move the down set up → we'll be left with down, right, left, and starting points sets
  - we move the right set left → we'll be left with up, right, down, and starting points sets


Center
Down
Right
Pole
Left
Up
Initial Points


Right + Up
Down + Left

- *The countable additivity property says that the sum of the measures of a countable family of pairwise disjoint measurable sets equals the measure of the union (stipulated to be measurable) of those sets.*
- *Lebesgue measure is not defined on a general metric spaces. If it were, in some special case such as the volume of a solid object, it would—by definition—satisfy the countable additivity property.*
  - ***Banach-Tarski Paradox fails on 1D and 2D because Lebesgue measure holds.***

Why is Banach-Tarski paradox so counterintuitive?

- It is always possible to carefully break a sphere into a finite number of sets such that when we take the union of those sets, we are left with two new spheres, each identical to our initial sphere.

- It is always possible to break a sphere into a finite number of non-measurable (not Lebesgue measurable) sets *where the additivity property of measure fails.* So that when we take the union of those sets, we are left with two new spheres, each identical to our initial sphere.
- ***Since we can prove it, why is Banach-Tarski paradox so counterintuitive?***
- Because it deals with non-measurable sets. A type of set that does not exist in the physical world we live in!

- *So our decomposition and reconstruction cannot be done in the physical world, because if you take and apple and cut it up into pieces with a knife, each of these pieces is “measurable.” That is, they have boundaries and a sense of volume, whereas the sets constructed using the Axiom of Choice do not obey these properties.*

The Axiom of Choice

- The Axiom of Choice: If we have some sets, we can always take one thing out of each set, put those things together, and create a new set! → very intuitive and obvious!
- Accepting The Axiom of Choice → Proves the existence of non-measurable sets (similar to what we just did over a sphere)
  - Every object with a volume could be broken into a group of sets, some of which are non-measurable sets

- *It is always possible to break an object into a finite number of non-measurable sets (where the additivity property of measure fails) such that when we take the union of those sets, we are left with some new objects, each identical to our initial object.*
- *In this way we could even turn an apple into the sun…*

References

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